Coordinate Changes for Integrals

IDEA: In Calculus I you used coordinate changes to solve

① $\int_{x=0}^{5} xe^{x^2}\,dx$ $\left(\begin{array}{l} u = x^2 \leftarrow \text{coordinate change, } \\ \underline{du = 2x\,dx} \text{ Parametrizes } \mathbb{R} \\ \quad\uparrow \end{array}\right.$

necessary differential composition

② Polar coordinate change

$$\iint_R e^{x^2+y^2}\,dA \rightarrow \iint_R re^{r^2}\,dA \qquad \left(\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right.$$

We want a more general way to compute these coordinate changes for integrals (make differential equations easier)

Answer: Jacobians!

Def// Jacobian of a coordinate change

$$\begin{cases} x_1 = x_1(u_1, u_2, \ldots, u_n) \\ x_2 = x_2(u_1, u_2, \ldots, u_n) \\ \vdots \\ x_n = x_n(u_1, u_2, \ldots, u_n) \end{cases}$$

$$\frac{d(x_1, x_2, \ldots, x_n)}{d(u_1, u_2, \ldots, u_n)} = \begin{bmatrix} dx_1/du_1 & dx_1/du_2 & \cdots & dx_1/du_n \\ dx_2/du_1 & dx_2/du_2 & \cdots & dx_2/du_n \\ \vdots & \vdots & \ddots & \vdots \\ dx_n/du_1 & dx_n/du_2 & \cdots & dx_n/du_n \end{bmatrix}$$

Example: Jacobian of polar coordinate change is

$$\frac{d(x,y)}{d(r,\theta)} = \det\begin{bmatrix} dx/dr & dx/d\theta \\ dy/dr & dy/d\theta \end{bmatrix}$$

$$= \cos\theta(r\cos\theta) - (\sin\theta(-r\sin\theta)) = r(\cos^2\theta + \sin^2\theta) = r$$

NB: if we reverse order of (r, θ), we get

$$\frac{\partial(x,y)}{\partial(\theta,r)} = \det \begin{bmatrix} dx/d\theta & dx/dr \\ dy/d\theta & dy/dr \end{bmatrix} = \det \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix}$$

$$= -r\sin\theta\cos\theta - r\cos\theta\cos\theta$$
$$= -r(\sin^2\theta + \cos^2\theta) = -r$$

Def// The (unsigned) Jacobian of a transformation is simply

$$\left| \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)} \right|$$

Prop: Let $f(x_1, x_2, \dots, x_n)$ be a function continuous on R and

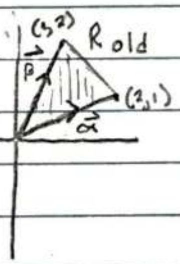
$$\begin{cases} x_1 = x_1(u_1, \dots, u_n) \\ \vdots \\ x_n = x_n(u_1, \dots, u_n) \end{cases}$$

Change by diff. fractions

$$\int_{R_{old}} f(x_1, \dots, x_n)\, dv_{old} = \int f(x_1(u_1, \dots, u_n), \dots, x_n(u_1, \dots, u_n)) \cdot \left| \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} \right| dv_{new}$$

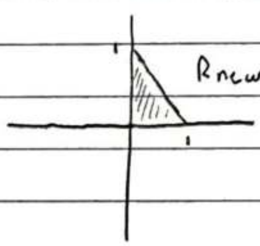
Example: compute $\iint_R (x-2y)\, dA$ for R the triangle w vertices $(0,0), (1,2), (2,1)$

$(x,y) = (u\vec{\alpha}, u\beta)$



Sol 1: using cartesian, split region and compute (do this on your own)

Sol 2: using a simple transformation



$(u,v) = (1,0) \to (x,y) = (2,1)$

$(u,v) = (0,1) \to (x,y) = (1,2)$

$$\begin{cases} x = 2u + v \\ y = u + 2v \end{cases}$$

Check that first triangle maps to second

Moreover, $R_{new} = \{(u,v): 0 \le u \le 1,\ 0 \le v \le 1-u\}$

$$\frac{d(x,y)}{d(u,v)} = \det \begin{bmatrix} dx/du & dx/dv \\ dy/du & dy/dv \end{bmatrix} = 4 - 1 = 3$$

$$\therefore \iint_R (x-2y)\, dA_{old} = \iint (2u+v) - 2(u+2) \cdot 3 \ dA_{new}$$

$$= 3\int_{u=0}^{1} \int_{v=0}^{1-u} -3v \ dv\, du$$

$$= -9 \int_0^1 \left(\tfrac{1}{2} v^2\right)\Big|_{v=0}^{1-u} du = -9/2 \int_{u=0}^{1} (1-u)^2$$

$$= 9/2 \left(1/3 \left[(1-u)^3\right]_{u=0}^{1} \right) = 3/2(-1) = -3/2$$

Generalizing Polar Coordinates To 3-Space

I) Cylindrical Coordinates

IDEA: Parametrize one plane w/ polar coordinates, leave orthogonal axis alone...

in particular, this coordinate change is

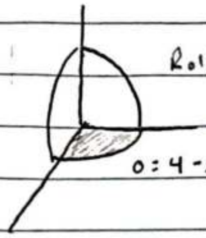
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

differential: $$\frac{d(x,y,z)}{d(r,\theta,z)} = \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta(r\cos\theta) + r\sin\theta(\sin\theta) + 0(r\cos^2\theta - r\sin^2\theta)$$
$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

Takeaway: $dA_{cartesian} = r\, dA_{cylindrical}$ for all cylindrical transformations

Example: Compute $\iiint_E (x+y+z)\,dv$, E in first octant, parabola $4-x^2-y^2=z$



Sol: parametrize cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad R_{new} = \left\{ (r,\theta,z) : \begin{array}{l} 0 \le r \le 2 \\ 0 \le \theta \le \pi/2 \\ 0 \le z \le 4-r^2 \end{array} \right.$$

$$\therefore \iiint_{E_{cart}} (x+y+z)\,dv = \iiint_{E_{polar}} (r\cos\theta + r\sin\theta + z)\, r\, dv$$

$$= \int_0^2 \int_0^{4-r^2} \int_0^{\pi/2} (r\cos\theta + r\sin\theta + z) r \, d\theta\, dz\, dr$$

$$= \int_{r=0}^{2} \int_{z=0}^{4-r^2} \Big[r\sin\theta - r\cos\theta + \theta z \Big]_{\theta=0}^{\pi/2}$$

$$= \int_{r=0}^{2} \int_{z=0}^{4-r^2} 2r^2 + \frac{\pi}{2} r z \, dz\, dr$$

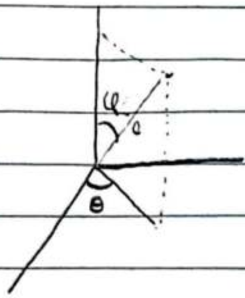
$$= \int_{r=0}^{2} 2r^2 z + \frac{\pi}{4} r z^2 \Big|_0^{4-r^2}$$

$$= \int_0^2 \left(8r^2 - 2r^4 + \pi/4 (16r - 8r^3 + r^5) \right) dr$$

$$= \left[\frac{8}{3} r^3 - \frac{2}{5} r^5 + \pi/4 \left(8r^2 - 2r^4 + \frac{1}{6} r^6 \right) \right]_{r=0}^{2}$$

$$= 64/3 - 64/5 + \pi 8/3$$

II Spherical coordinates: Every point in $\mathbb{R}^3$ lives on a sphere

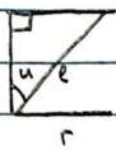


we parametrize via

ρ = distance from (x, y, z) to origin

θ = angle from x axis to point $(x, y, 0)$

φ = angle from y axis to point (x, y, z)



$$\begin{cases} x = r\cos\theta \rightarrow \rho\sin(\varphi)\cos\theta \\ y = r\sin\theta \rightarrow \rho\sin(\varphi)\sin\theta \\ z = r\cos(\varphi) \end{cases}$$